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Geometry on the Cubic Scroll of the Second Kind.

BY FREDERICK C. FERRY.

It is the object of the present paper to give a detailed treatment of several of the more interesting questions of the geometry on the cubic scroll of the second kind, and especially to consider the surfaces which can be passed through any curve on this scroll, so far as the orders of those surfaces and the natures of the residual intersections are concerned. Among published articles, those bearing most directly on the subject proposed are perhaps the following:

Clebsch, "Die Geometrie auf den Flächen dritter Ordnung." Kronecker J., LXV., 359–380.

Chasles, "Ueber die Steiner'sche Fläche." Kronecker J., LXVII., 1–22.

Chasles, "Bemerkung über die Geometrie auf den windschiefen Flächen dritter Ordnung." Math. Ann., I., 634–636.

Voss, "Zur Theorie der windschiefen Flächen." Math. Ann., VIII., 54–135.

Halphen, "Sur la classification des Courbes algébriques." J. de l'Ec. Pol., LII.

Noether, "Zur Grundlegung der Theorie algebraischen Raumcurven." Berlin, 1883.

Sturm, "Ueber die Curven auf der allgemeinen Fläche dritter Ordnung." Klein, Math. Ann., XXI., 457–515.

Rohn, "Die Raumcurven auf den Flächen dritter Ordnung." Leipz. Ber., XLVI., 84–119.

And, in general, the methods employed and the results obtained in the following pages will be found to be analogous to those presented in a paper by the present writer on the "Geometry on the Cubic Scroll of the First Kind," published in the Archiv for Mathematik og Naturvidenskab, B. XXI., Nr. 3.

I.—*The Cubic Scroll of the Second Kind.*

The equation in homogeneous coördinates of the cubic scroll of the second kind is

$$y^3 - x(ys + xz) = 0.$$

Let this equation, or the surface itself, as the case may demand, be denoted by Σ. The double line on Σ is given by $x = 0$, $y = 0$; the plane $x = 0$ touches Σ along the entire length of the double line, and hence contains that line three times; every other plane through the double line contains that line twice, and every plane through the double line cuts out a generator from Σ, the generator cut out by the plane $x = 0$ being coincident with the double line. The plane $y = 0$ is not a determinate plane, for the substitution of $y + \theta x$ for y, with a corresponding change in the coördinates z and s, leaves Σ unchanged in form. The planes $z = 0$ and $s = 0$ change with the plane $y = 0$ and are not determined when the plane $y = 0$ is determined, for the substitution of $z + \lambda x$ and $s + \lambda y$ for z and s respectively leaves Σ unchanged in form. The tangent planes to Σ at any point of the double line are $x = 0$ and the tangent plane to the hyperboloid $ys + xz = 0$ at that point; these two tangent planes coincide at the point $x = 0$, $y = 0$, $s = 0$, hence, this point is a pinch-point on Σ.

Any generator on Σ is given by equations of the form $y = \alpha x$, $z = \alpha^3 x - \alpha s$; the generator given by $y = \alpha_1 x$, $z = \alpha_1^3 x - \alpha_1 s$ meets the double line at its point of intersection with the plane $z + \alpha_1 s = 0$, hence, the points $z = -\alpha s$ on the double line correspond to the planes $y = \alpha x$ through the double line, and in particular the point $s = 0$ on the double line corresponds to the plane $x = 0$ through the double line; i. e., the pinch-point corresponds to the plane which is tangent to Σ all along the double line; thus, to each point of the double line corresponds a generator through that point, while to the point $s = 0$ of the double line corresponds a generator coincident with the double line itself. Therefore, it may be said that two generators on Σ meet the double line at every point, of which one coincides with the double line itself, while at the pinch-point both generators coincide with the double line.

To distinguish the two sheets of Σ in the neighborhood of the double line that will be called the first sheet in which the generators at successive points of the double line are distinct, and the other, in which the generator at every point of the double line coincides with the double line, will be called the second sheet.

II.—*Coördinates on* Σ.

Assuming $x/y = \lambda/\mu$ and $y/z = \lambda/\nu$ and inserting these values in Σ, there results $s/y = \dfrac{\mu^2 - \lambda\nu}{\lambda\mu}$, whence the coördinates λ, μ, ν on Σ are connected with the coördinates x, y, z, s on Σ by the relations

$$x : y : z : s = \lambda^2 : \lambda\mu : \mu\nu : \mu^2 - \lambda\nu .$$

Along the double line in the first sheet, ν/λ and ν/μ take infinite values, while λ/μ is finite, and any generator is given by an equation of the form $a\lambda + b\mu = 0$; hence, any homogeneous equation in λ and μ of degree n represents n generators lying in the first sheet in the neighborhood of the double line; and, in particular, $\lambda = 0$ gives the generator at the pinch-point, and $\mu = 0$ the generator which is cut from Σ by the plane $y = 0$. Given, then, a homogeneous equation in λ, μ, ν representing a curve on Σ, to determine the points where this curve or branches of this curve lying in the first sheet in the neighborhood of the double line meet the double line, it is necessary only to put $\nu/\lambda = \nu/\mu = \infty$ or $\lambda/\nu = \mu/\nu = 0$ in the given equation, whence there will result a homogeneous equation in λ and μ giving the generators at the points desired.

The double line in the second sheet is given by $\lambda = 0$, and to its points correspond finite values of the ratio μ/ν; thus, $a\mu + b\nu = 0$ gives a point on the double line in the second sheet; and, given a homogeneous equation in λ, μ, ν representing a curve on Σ, to determine the points where this curve or branches of this curve lying in the second sheet in the neighborhood of the double line meet the double line, it is necessary only to put $\lambda = 0$ in the given equation and to solve the resulting equation in μ and ν.

Any plane $ax + by + cz + ds = 0$ cuts from Σ a cubic curve whose equation in λ, μ, ν is found by direct substitution to be

$$a\lambda^2 + b\lambda\mu + c\mu\nu + d(\mu^2 - \lambda\nu) = 0 .$$

This curve is found by the methods given above to meet the double line in the first sheet where that line is met by the generator $c\mu - d\lambda = 0$ and to meet the double line in the second sheet at the point determined by the equation $c\nu + d\mu = 0$; now these two points, one lying in the first sheet and the other in the second sheet, are known to be coincident since they are both cut from the double line by the same plane; hence, to the point $d\mu + c\nu = 0$ on the double line in the second sheet corresponds by coincidence that point on the double line in the first sheet which lies on the generator $d\lambda - c\mu = 0$. So, in general, the change of μ to λ and of ν to $-\mu$ in the equation determining a point or points on the double line in the second sheet gives the equation of the generator or generators passing through the same point or points on the double line regarded as lying in the first sheet. Thus the point on the double line in the second sheet given by $\mu = 0$ lies at the pinch-point in the first sheet, and the point on the double line in the second sheet given by $\nu = 0$ lies at the point where the generator $\mu = 0$ meets the double line in the first sheet. In general, a point on the double line will be said to lie in the first sheet or the second sheet according as the equation by which it is determined is of the form $a\lambda + b\mu = 0$ or $a\mu + b\nu = 0$; thus, a homogeneous equation in λ and μ of degree n determines n points on the double line in the first sheet of Σ, and, similarly, a homogeneous equation of degree n in μ and ν determines n points on the double line in the second sheet of Σ.

III.—*Curves on* Σ.

It is evident that, in general, proper curves on Σ are given by irreducible homogeneous equations in λ, μ, ν. Let such an equation, or the curve represented thereby, as the case may demand, be denoted by ϕ; let the degree of ϕ in all three variables be denoted by p and its degree in the variable ν by q, whence it follows that $p \geqq q$. If the terms of ϕ be arranged according to the powers of ν, thus:

$$\phi \equiv U_p + \nu U_{p-1} + \nu^2 U_{p-2} + \ldots\ldots + \nu^\kappa U_{p-\kappa} + \ldots\ldots + \nu^q U_{p-q} = 0,$$

where $U_{p-\kappa}$ denotes a homogeneous polynomial in λ and μ of degree $p - \varkappa$, then, as has been shown, $U_{p-q} = 0$ must give at once the $p - q$ generators meeting the double line in the points where that line is met by the curve ϕ in the first sheet; hence, the curve ϕ has $p - q$ points of intersection with the double line in the first sheet of Σ.

To find the points of intersection of the curve ϕ with any generator $a\lambda + b\mu = 0$, it is necessary to substitute $-b/a \,.\, \mu$ for λ in ϕ; having performed this substitution, the equation is divisible by a factor μ^{p-q}; this factor having been rejected, there is left an equation of degree q in μ and ν, which determines q points where the curve ϕ meets the generator in question. Hence, ϕ has q points of intersection with any generator; thus, ϕ has q points of intersection with the double line in the second sheet, which points are found by equating λ

to zero in ϕ and solving the resulting equation in μ and ν, after rejecting the factor μ^{p-q}. If ϕ be arranged according to powers of λ, thus:

$$\phi \equiv V_p + \lambda V_{p-1} + \lambda^2 V_{p-2} + \dots + \lambda^\theta V_{p-\theta} + \dots + \lambda^\iota V_{p-\iota} = 0, \quad (\iota \leqq p),$$

where $V_{p-\theta}$ denotes a homogeneous polynomial in μ and ν of degree $p - \theta$, then will $V_p \equiv \mu^{p-q}$. V_q' and $V_q' = 0$ gives the q points where the double line is met by the curve ϕ in the second sheet. Since, then, the curve ϕ meets the double line in $p - q$ points in the first sheet and in q points in the second sheet, it has in all $p - q + q = p$ points of intersection with the double line regarded as lying in both sheets. Any plane through the double line of Σ contains, in addition to that line, a generator of Σ; such a plane meets the curve ϕ in p points on the double line and in q points on the generator in question, and hence contains $p + q$ points of intersection with the curve; hence, the order of the curve ϕ is $p + q$, which will be denoted by m, so that always $m = p + q$. The curve ϕ will often be designated as a (p, q), where p and q have the meanings assigned above. Thus any generator is a $(1, 0)$, the double line regarded as lying in the first sheet is a $(0, 1)$ and regarded as lying in the second sheet is a $(1, 0)$ like any other generator, while, considered as lying in both sheets, it may be said to be a $(1, 1)$, a conic. These results agree entirely with those given in the consideration of the geometry on the cubic scroll of the first kind, if Σ be regarded as obtained from that scroll by allowing the linear director to tend to coincide with the double line in the one sheet, while a generator tends to coincide with the double line in the other sheet, the former coincidence determining the first sheet of Σ and the latter the second sheet of Σ; thus, on the cubic scroll of the first kind, each generator is a $(1, 0)$, the linear director is a $(0, 1)$, and the double director a $(1, 1)$.

Two curves, (p, q) and (p', q'), are said to belong to the same *species* when $p = p'$ and $q = q'$. The number of distinct species of curves of any order m is evidently the greatest integer in $m/2$, the species $(p, 0)$, consisting in each case of only a certain number of generators, being omitted.

A curve ϕ will be said to have a *pair* of branches at any point of the double line when two of its branches lying, one in the one sheet and the other in the other sheet in the neighborhood of the point, intersect at that point of the double line. With reference to the arrangements of the terms of ϕ, according to powers of ν and of λ respectively, as given above, the condition that ϕ have a pair of branches at the point where any generator $a\lambda + b\mu = 0$ meets the double line is that $V_p \equiv (a\mu - b\nu) . V'_{p-1}$ when $U_{p-q} \equiv (a\lambda + b\mu) . U'_{p-q-1}$, and that ϕ have a pair of branches wherever it meets the double line, it is necessary and sufficient that

$$V_p \equiv C_1 (a_1\mu - b_1\nu)^{\alpha_1} . (a_2\mu - b_2\nu)^{\alpha_2} \ldots . (a_\tau\mu - b_\tau\nu)^{\alpha_\tau} . \mu^{p-q},$$

when $$U_{p-q} \equiv C_2 (a_1\lambda + b_1\mu)^{\beta_1} . (a_2\mu + b_2\mu)^{\beta_2} \ldots . (a_\tau\lambda + b_\tau\mu)^{\beta_\tau},$$

where $\alpha_r \geqq 1$, $\beta_r \geqq 1$, $\alpha_1 + \alpha_2 + \ldots . + \alpha_\tau = q$ and $\beta_1 + \beta_2 + \ldots . \beta_\tau = p - q$. Here can occur several pairs of branches at any point where ϕ meets the double line, and superfluous branches in addition to the number sufficient to make up the pair or pairs of branches at any point can occur. If $p = 2q$, just as many of these superfluous branches will occur in the one sheet as in the other; and if, further, $\alpha_r = \beta_r$ while $p = 2q$, then will only pairs of branches be found without superfluous branches; while in this case, if $\alpha_r = \beta_r = 1$, these pairs will all occur singly. It is geometrically evident that $p = 2q$ when only pairs without superfluous branches occur, or when as many superfluous branches occur in the one sheet as in the other, since in either case just as many points of the curve must lie on the double line in the one sheet as in the other, demanding that $p - q = q$.

If $U_{p-q} \equiv (a\lambda + b\mu)^2 . U'_{p-q-2}$, then will two branches of ϕ, both lying in the first sheet in the neighborhood of the double line, intersect the double line at the same point, but without forming thereby a pair of branches as that term has been defined. Similarly, if

$$U_{p-q} \equiv (a\lambda + b\mu)^\alpha . U'_{p-q-\alpha},$$

then will α branches, all lying in the first sheet in the neighborhood of the double line, intersect that line at the same point. Likewise, if

$$V_p \equiv (c\mu + d\nu)^\beta . V'_{p-\beta},$$

then will β branches of ϕ, all lying in the second sheet in the neighborhood of the double line, intersect that line at the same point.

That every point where ϕ meets the double line be a multiple point of ϕ, resulting from the intersection on that line of branches lying either in the same or in different sheets of Σ in the neighborhoods in question, it is necessary and sufficient that

$$U_{p-q}\overline{V}_p \equiv C . \lambda^{p-q} . (a_1\lambda + b_1\mu)^{\beta_1} . (a_2\lambda + b_2\mu)^{\beta_2} \ldots . (a_\sigma\lambda + b_\sigma\mu)^{\beta_\sigma},$$

where $\beta_r \geqq 2$ and $\overline{V}_p$ is what V_p becomes when μ and ν have been changed to λ and $-\mu$ respectively wherever they occur in it. If $\beta_r = 2$, each point where ϕ meets the double line will be a double point of ϕ, in which case p is evidently even.

If $U_{p-q} \equiv \lambda^{\alpha} . U'_{p-q-\alpha}$, then will ϕ pass through the pinch-point α times, while each of the α branches lies in the first sheet in the neighborhood of the pinch-point. Similarly, if $V_p \equiv \mu^{p-q+\beta} . V'_{q-\beta}$, then will ϕ pass through the pinch-point β times, while each of the β branches lies in the second sheet in the neighborhood of the pinch-point. And, in general, the curve ϕ has the pinch-point for a point of multiplicity γ when, and only when, $U_{p-q} . \overline{V}_p \equiv \lambda^{p-q+\gamma} . U'_{p-\gamma}$ and the point where any generator $a\lambda + b\mu = 0$ meets the double line is a point of multiplicity γ on the curve ϕ when, and only when,

$$U_{p-q} . \overline{V}_p \equiv \lambda^{p-q} . (a\lambda + b\mu)^{\gamma} . U'_{p-\gamma} .$$

Unless otherwise stated, it will be supposed henceforth that every equation ϕ employed is homogeneous and irreducible.

IV.—*The Curve ϕ as the Intersection, Total or Partial, of Σ with a Surface S.*

To find the equation of a surface S which shall cut the curve ϕ from Σ, it is necessary to substitute x, y, z and s for λ^2, $\lambda\mu$, $\mu\nu$ and $\mu^2 - \lambda\nu$ respectively in the equation ϕ, or in $\omega\phi$, where ω is such a factor as shall render this substitution possible. Evidently, this factor ω must be homogeneous in λ, μ, ν, and let its degree be denoted by n', while m' shall denote the degree of the equation $\omega\phi$ in x, y, z, s after the required substitution has taken place; hence, m' will denote the order of the surface S. If n'_1 denote the degree of ω in the variable ν, then will a residual intersection of order at least as great as $n' + n'_1$ result from the intersection of S and Σ, and, therefore, that the curve ϕ be the complete intersection of S and Σ, it is necessary that $n' + n'_1 = 0$; i. e., it must be true at once that

$$\phi \equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2 - \lambda\nu) = 0 ,$$

so that the required substitution may be immediately possible. Since in this case ν enters into ϕ only in one or both of the combinations $\mu\nu$ and $\mu^2 - \lambda\nu$, and since one or both of these combinations must make up a term $C(\mu\nu)^{\alpha} . (\mu^2 - \lambda\nu)^{\beta}$ of ϕ where $\alpha + \beta = 1/2 . p$ (else would ϕ be reducible by the factor λ), it is clear that in every case of complete intersection $p=2q$. Again, when

$$\phi\equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2-\lambda\nu)=0,$$

the intersections of ϕ with the double line in the first sheet are given by an equation which is obtained directly from the above and may be called $f'(\mu\nu-\lambda\nu)/\nu^{\frac{p}{2}}=0$; similarly, the intersections of ϕ with the double line in the second sheet are given by an equation $f''(\mu\nu, \mu^2)/\mu^{\frac{p}{2}}=0$; if the divisions by ν and μ be performed, these become respectively $f'(\mu, -\lambda)=0$ and $f''(\nu, \mu)=0$; but, since these two equations are made up of the same terms, it is clear that $f'\equiv f''$; it has been shown that two equations $f(\mu, -\lambda)=0$ and $f(\nu, \mu)=0$ give the same points of the double line; hence, when ϕ is capable of immediate substitution, not only must $p=2q$, but also every point where ϕ meets the double line must be a multiple point of ϕ resulting from the occurrence there of a pair or pairs of branches without superfluous branches; the same is geometrically evident whenever the intersection is complete. In general, $m'=1/2.(p+n')$, since the required substitution changes a quadratic to a linear factor in every case; hence, when $n'=0$, it follows that $m'=1/2.p$; and, since

$$\tfrac{3}{2}p=p+\tfrac{1}{2}p=p+q=m,$$

no residual intersection can appear, and the curve ϕ is the total intersection of a surface S of order $\frac{1}{2}p$ with Σ when, and only when,

$$\phi\equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2-\lambda\nu)=0.$$

Thus no curve (p, q), where p is odd, can be the complete intersection of any surface S with Σ. This also is at once evident geometrically.

If
$$\phi\not\equiv f(\lambda^2, \lambda\mu, \mu\nu, \mu^2-\lambda\nu)=0,$$

it will be necessary to multiply ϕ by such a factor ω of degree n' in λ, μ, ν and n'_1 in ν, where $n'\geqq 1$, as shall render the required substitution possible. The curve ϕ in this case will not be the complete intersection of any surface S with Σ, but there will result a residual intersection of order at least as great as $n'+n'_1$. The order of S being $m'=\frac{1}{2}(p+n')$, m' will be a minimum when n' is a minimum; and those surfaces S of the lowest possible orders, cutting the curves ϕ from Σ, are of special interest, and will be given particular attention. Since, as has been noticed, the required substitution replaces a quadratic by a linear factor in every case, it is clear that n' must be even when p is even and odd when p is odd.

Since ν can be replaced in $\omega\phi$ only in the combinations $\mu\nu$ and $\lambda\nu$ (the latter occurring in the form $\mu^2 - \lambda\nu$), it is clear that no term of $\omega\phi$ can contain ν to a power greater than the sum of the powers of λ and μ in that term, and this will be true especially of those terms of $\omega\phi$ coming from terms in ϕ which contain the greatest, i. e., the q^{th} power of ν; hence, if $p < 2q$, it will be necessary that ω contain in each of its terms λ or μ, or both, to a power or sum of powers at least as much greater than the degree of ω in ν as is the excess of q over $p - q$; i. e., it must always be true that $p + n' \geqq 2(q + n_1')$; since $m' = \frac{1}{2}(p + n')$, it is now seen that $m' \geqq q + n_1'$, and hence the lowest possible order of S is at least as great as q in every case; it has already been found to be q in the case where (p, q) is the total intersection of S and Σ.

If ω involves ν, then will ω contribute to the residual intersection of S and Σ an irreducible curve of order at least as great as two; while, if ω does not involve ν, it can bring into the residual intersection only generators, n' in number. Therefore, unless otherwise stated, it will be supposed that ω is a homogeneous function involving only λ and μ, and consequently representing n' generators.

Having chosen, then, ω_τ for any given curve ϕ in such a way that $p + \tau \geqq 2q$ and *even*, every term of $\omega_\tau\phi = 0$ can be at once resolved into a constant times a product of powers of $\lambda\nu$, $\mu\nu$, $\lambda\mu$, λ^2 and μ^2; and, further, since

$$\begin{aligned}(\lambda\nu)^\alpha.(\mu^2)^\beta &= (\mu\nu)^{2\beta}.(\lambda\nu)^{\alpha-2\beta}.(\lambda^2)^\beta && \text{if } \alpha \geqq 2\beta \\ &= (\mu\nu)^\alpha.(\lambda\mu)^{2\beta-\alpha}.(\lambda^2)^{\alpha-\beta} && \text{if } 2\beta \geqq \alpha \geqq \beta \\ &= (\mu\nu)^\alpha.(\lambda\mu)^\alpha.(\mu^2)^{\beta-\alpha} && \text{if } \beta \geqq \alpha,\end{aligned}$$

it follows that this factoring of the terms $\omega_\tau\phi = 0$ can always be done in such a way that no term contains both $\lambda\nu$ and μ^2 as factors. When $\omega_\tau\phi = 0$ has been thus factored in each term, the substitution of x, y, z for λ^2, $\lambda\mu$, $\mu\nu$ respectively, as many times in each term as each of the factors in question occurs there, is at once possible; and thus the equation $\omega_\tau\phi = 0$ is made to involve the variables λ, μ, ν only in the combinations $\lambda\nu$ and μ^2. Since s is to be inserted in place of $\mu^2 - \lambda\nu$ by the required substitution, it is permissible, as a step toward that end, to replace $\lambda\nu$ wherever it occurs in the factored form of $\omega_\tau\phi = 0$ by $\mu^2 - s$, so that any factor $(\lambda\nu)^\gamma$ now becomes

$$(\lambda\nu)^\gamma = (\mu^2 - s)^\gamma = (\mu^2)^\gamma + C_1 (\mu^2)^{\gamma-1} s + \ldots\ldots$$
$$+ C_\kappa (\mu^2)^{\gamma-\kappa} . s^\kappa + \ldots\ldots + C_1 \mu^2 s^{\gamma-1} + s^\gamma,$$

containing only s and powers of μ^2. Thus is $\omega_\tau \phi = 0$ made to involve only x, y, z, s and μ^2, and in this form, arranged according to powers of μ^2, it may be written thus:

$$\overline{\omega_\tau \phi} = (\mu^2)^{\frac{1}{2}(p+\tau)} . \psi_0 + (\mu^2)^{\frac{1}{2}(p+\tau)-1} . \psi_1 + \ldots\ldots$$
$$+ (\mu^2)^{\frac{1}{2}(p+\tau)-\kappa} . \psi_\kappa + \ldots\ldots + \mu^2 . \psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0,$$

where ψ_κ represents a homogeneous polynomial in x, y, z, s of degree $\varkappa$.

Let θ_1 and θ_2 denote the number of times that λ and μ respectively occur as factors of all the terms of ω_τ, so that $\omega_\tau \equiv \lambda^{\theta_1} . \mu^{\theta_2} . \omega'_{\tau-\theta_1-\theta_2}$; and let it be assumed for the present that $\theta_1 = 0$. Since the variables x and y can occur in the above equation only from the substitution of the same for the combinations λ^2 and $\lambda\mu$ respectively, it is clear that that equation can contain neither of those variables as a factor unless the equation $\omega_\tau \phi = 0$ contain λ as a factor; that case has now been excluded, for by supposition ϕ is irreducible and $\theta_1 = 0$. Similarly, since the variables z and s enter above only from the substitution of z and $\mu^2 - s$ respectively for $\mu\nu$ and $\lambda\nu$, it follows that the above equation $\overline{\omega_\tau \phi} = 0$ can have neither z nor s as a factor; for ν is not a factor of all the terms of ϕ and does not occur at all in ω_τ. If μ^2 occur ρ times as a factor of the equation in question, it may be removed by dividing by $\mu^{2\rho}$ and making a corresponding reduction in θ_2. But such a limit can be found for θ_2 as shall make the occurrence of μ^2 thus as a factor of this equation impossible, and in the following manner: Among the terms of ϕ is one at least without μ, and consequently of the form $C . \lambda^{p-\alpha} . \nu^\alpha$ where $0 \leqq \alpha \leqq q$; such a term, when multiplied by the term of ω_τ, which is of the lowest degree in μ and may be expressed as $C' . \lambda^{\tau-\theta_2} . \mu^{\theta_2}$, will lead to one of the terms of lowest degree in μ^2 in the derived equation; and, if the factoring and substitution be performed thus:

$$C . \lambda^{p-\alpha} . \nu^\alpha . C' . \lambda^{\tau-\theta_2} . \mu^{\theta_2} \equiv C'' . (\mu\nu)^\alpha . (\lambda\mu)^{p+\tau-\alpha-\theta_2} . (\mu^2)^{\theta_2 - \frac{1}{2}(p+\tau)} \equiv (\mu^2)^{\theta_2-\frac{1}{2}(p+\tau)} . \psi_{p+\tau-\theta_2},$$

if $\theta_2 \geqq \frac{1}{2}(p+\tau)$; or thus:

$$\equiv C'' . (\mu\nu)^\alpha . (\lambda\mu)^{\theta_2-\alpha} . (\lambda^2)^{\frac{1}{2}(p+\tau)-\theta_2} \equiv \psi'_{\frac{1}{2}(p+\tau)},$$

if $\alpha \leqq \theta_2 \leqq \frac{1}{2}(p+\tau)$; or thus:

$$\equiv C'' . (\mu\nu)^{\theta_2} . (\lambda\nu)^{\alpha-\theta_2} . (\lambda^2)^{\frac{1}{2}(p+\tau)-\alpha-\theta_2} \equiv \psi''_{\frac{1}{2}(p+\tau)},$$

if $\theta_2 \leqq \alpha$; and only in the first case does the term in question give rise to a term or group of terms containing μ^2 as a factor; therefore, a term which shall not contain the factor μ^2 will always be found in $\overline{\omega_\tau \phi} = 0$ if $\theta_2 \leqq \frac{1}{2}(p+\tau)$; and, in general, if $\theta_2 = \frac{1}{2}(p+\tau)+\gamma$, then will $(\mu^2)^\gamma$ be a factor of every term of the equation $\overline{\omega_\tau \phi} = 0$. Unless otherwise stated, it will be supposed henceforth that $\theta_2 \leqq \frac{1}{2}(p+\tau)+1$.

Among the terms of ϕ is one, at least, without λ and, consequently, of the form $C.\mu^{p-\alpha}.\nu^\alpha$, where $0 \leqq \alpha \leqq q$. Such a term, when multiplied by that term of ω_τ which is of the highest degree in μ and may be written $C'.\mu^\tau$, when the required substitution is performed, gives

$$C.\mu^{p-\alpha}.\nu^\alpha.\,C'.\mu^\tau = C''.(\mu\nu)^\alpha.(\mu^2)^{\frac{1}{2}(p+\tau)-\alpha} = C''.(\mu^2)^{\frac{1}{2}(p+\tau)-\alpha}.z^\alpha.$$

One or more such terms must always occur in the equation $\omega_\tau \phi = 0$. All the terms in ϕ which are free from λ, together determine, as has been seen, the points where the curve ϕ meets the double line in the second sheet; evidently, the lowest value which α takes in these terms is the same as the order of the multiplicity on the curve ϕ of the point $\lambda = 0$, $\mu = 0$, if that multiplicity have reference here only to the order of intersection at the given point of the branches of ϕ lying in the second sheet in the neighborhood of that point; i. e., if the smallest value which α, as here defined, takes in the equation of the given curve ϕ be denoted by α_1, then will α_1 give the number of times the curve ϕ passes through the point where the generator $\mu = 0$ meets the double line, that point being here regarded as lying in the second sheet. Hence, the equation $\overline{\omega_\tau \phi} = 0$ may always be written thus:

$$\begin{aligned}\overline{\omega_\tau \phi} \equiv (\mu^2)^{\frac{1}{2}(p+\tau)-\alpha_1}.z^{\alpha_1}.\psi_0 + (\mu^2)^{\frac{1}{2}(p+\tau)-1}.\psi_1 + \cdots\cdot \\ + (\mu^2)^{\frac{1}{2}(p+\tau)-\kappa}.\psi_\kappa + \cdots\cdot + \mu^2.\psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0,\end{aligned}$$

where the first term and one, at least, of the last two terms must occur. Since ϕ involves the variable ν in some term or terms to the power q, but no higher power of that variable, and since ω_τ does not involve ν at all, it follows that $\overline{\omega_\tau \phi} = 0$ has likewise in some term or terms the variable ν to the power q, but contains no higher power of that variable. Those terms in $\overline{\omega_\tau \phi} = 0$ involving the q^{th} power of ν have as factors, when that equation has been put in the factored form for substitution, the expression or expressions of the form $(\lambda\nu)^\rho.(\mu\nu)^\sigma$ where $\rho + \sigma = q$; these give rise to terms in $\overline{\omega_\tau \phi} = 0$ involving $s^\rho z^\sigma$, and, consequently, the equation $\overline{\omega_\tau\phi} = 0$ is of degree q in the variables z, s in some term or terms, but of no higher degree in those variables in any term. Therefore, $\psi_{\frac{1}{2}(p+\tau)-\kappa}$ is of degree at least as great as $\frac{1}{2}(p+\tau) - q - \kappa$ in the variables x, y in every one of its terms.

The equation $\Sigma \equiv y^3 - x(ys + xz) = 0$ may be written $y^3 = x\pi$ if $\pi \equiv ys + xz$. Since the substitution of y^3 for $x\pi$ or of $x\pi$ for y^3 in any or all the terms of the equation of any surface S which cuts the curve ϕ from Σ, is equivalent to replacing S by another surface S' whose equation $S' \equiv S + \psi'_{m'-3} \cdot \Sigma$, where S' contains the entire intersection of S and Σ and is of the same order as S, it is clear that such interchange of y^3 and $x\pi$ in any of the equations now under consideration is allowable; and this means of effecting a change in the equations in question will frequently be employed without further explanation.

Under the conditions of the required substitution, any even power of μ^2 as, e. g., $(\mu^2)^{2\eta}$, can be expressed thus:

$$\begin{aligned}(\mu^2)^{2\eta} &= (\mu^2)^\eta . (s + \lambda\nu)^\eta = (\mu^2 s + \lambda\mu^2\nu)^\eta = (\mu^2 s + yz)^\eta \\ &= (\mu^2)^\eta . s^\eta + \eta . (\mu^2)^{\eta-1} . s^{\eta-1} . yz + \dots + \eta . \mu^2 s . y^{\eta-1} z^{\eta-1} + y^\eta z^\eta,\end{aligned}$$

which, if $\eta \geqq 3$, can be expressed otherwise as

$$\begin{aligned}(\mu^2)^{2\eta} = (\mu^2)^\eta s^\eta + \eta . (\mu^2)^{\eta-1} s^{\eta-1} . yz + \dots \\ + \eta \frac{(\eta-1)}{2} . (\mu^2)^2 s^2 . y^{\eta-2} z^{\eta-2} + \frac{\pi}{\lambda^2} x y^{\eta-3} z^{\eta-1} (\eta ys + xz).\end{aligned}$$

Similarly, any odd power of μ^2 as, e. g., $(\mu^2)^{2\eta+1}$, can be expressed thus:

$$(\mu^2)^{2\eta+1} = (\mu^2)^{\eta+1} s^\eta + \eta . (\mu^2)^\eta s^{\eta-1} . yz + \dots + \eta . (\mu^2)^2 s . y^{\eta-1} z^{\eta-1} + \mu^2 . y^\eta z^\eta,$$

which, if $\eta \geqq 2$, can be put thus:

$$(\mu^2)^{2\eta+1} = (\mu^2)^{\eta+1} s^\eta + \eta . (\mu^2)^\eta s^{\eta-1} . yz + \dots + \eta . (\mu^2)^2 s . y^{\eta-1} z^{\eta-1} + \frac{\pi}{\lambda^2} x y^{\eta-1} z^\eta .$$

By the application of this method of reduction the necessary number of times, any power of μ^2, as, e. g., $(\mu^2)^\theta$. where $\theta \geqq 3$, can be brought into the form

$$(\mu^2)^\theta = (\mu^2)^{\gamma_1} . s^{\theta-\gamma_1} + (\mu^2)^4 . \psi'_{\theta-4} + (\mu^2)^3 . \psi'_{\theta-3} + (\mu^2)^2 . \psi'_{\theta-2} + \frac{\pi}{\lambda^2} . \psi'_{\theta-1},$$

where γ_1 has the value 3 or 4 according as $\theta = 2^\kappa + \rho_1$ or $2^\kappa + \rho_2$, while $\rho_1 = 1, 2, 3, \dots 2^{\kappa-1}$ and $\rho_2 = 0, 2^{\kappa-1} + 1, 2^{\kappa-1} + 2, \dots, 2^\kappa - 1$; $\psi'_{\theta-r}$ denotes a homogeneous polynomial in x, y, z, s of degree $\theta - r$ in all four variables together and of degree at least as great as unity in x, y.

Applying this method of reduction to $\overline{\omega_\tau\phi} = 0$, it is evident that, whenever $\frac{1}{2}(p+\tau) - \alpha_1 \geqq 3$, the term $(\mu^2)^{\frac{1}{2}(p+\tau)-\alpha_1} z^{\alpha_1}.\psi_0$ gives rise to a term or series of terms which, arranged according to descending powers of μ^2, begin thus:

$$(\mu^2)^{\gamma_1}.\, s^{\frac{1}{2}(p+\tau)-\alpha_1-\gamma_1}.\, z^{\alpha_1}.\, \psi_0 + \ldots .$$

and may be written in the form

$$(\mu^2)^{\gamma_1}.\, (z,\, s)^{\frac{1}{2}(p+\tau)-\gamma_1} + \ldots .$$

and the entire equation thus finally takes the form

$$\overline{\overline{\omega_\tau\phi}} \equiv (\mu^2)^{\gamma_1}.\, (z,\, s)^{\frac{1}{2}(p+\tau)-\gamma_1} + (\mu^2)^4.\, \psi''_{\frac{1}{2}(p+\tau)-4} + (\mu^2)^3\, \psi''_{\frac{1}{2}(p+\tau)-3}$$
$$+ (\mu^2)^2.\, \psi''_{\frac{1}{2}(p+\tau)-2} + \frac{\pi}{\lambda^2}\cdot \psi''_{\frac{1}{2}(p+\tau)-1} + \mu^2.\, \psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0$$

The terms making up any one of the groups designated by $\psi''_{\frac{1}{2}(p+\tau)-r}$ may be of two kinds: first, those obtained by the reduction of terms of $\overline{\omega_\tau\phi} = 0$ which involve $(\mu^2)^\theta$, where $\theta \geqq 5$; from the formulæ above it is clear that all terms entering thus in $\psi''_{\frac{1}{2}(p+\tau)-1}$ must be of degree at least as great as two in x, y, and all terms entering thus in $\psi''_{\frac{1}{2}(p+\tau)-2}$ must be of degree at least as great as unity in x, y; and, secondly, in any group $\psi''_{\frac{1}{2}(p+\tau)-r}$, where $r > 1$, there will be included the terms of $\psi_{\frac{1}{2}(p+\tau)-r}$ without change therein; terms in $\psi''_{\frac{1}{2}(p+\tau)-2}$ coming from $\psi_{\frac{1}{2}(p+\tau)-2}$ are known to be of degree as great as $\frac{1}{2}(p+\tau) - q - 2$ in x, y; and this will be as great as unity whenever $\frac{1}{2}(p+\tau) - q \geqq 3$. The first group of terms and one, at least, of the last two groups of terms in $\overline{\overline{\omega_\tau\phi}} = 0$, as written above, will in every case actually occur, and γ_1 can have one or both the values 3 and 4 but no other; in case γ_1 takes both the values 3 and 4, the first group may be broken up, according to the third and fourth powers of μ^2 then occurring, into two groups of terms.

From the equations of substitution, $y^3 = x\pi$, $\lambda^2 = x$, $\lambda\mu = y$, etc., it is easily found that

$$\begin{aligned}
\lambda^2\mu^2 &= y^2, \\
\lambda^2\mu^4 &= \lambda^2 \quad (\mu^2 s + yz) = y^2 s + xyz = y\pi, \\
\lambda^2\mu^6 &= \lambda^2\mu^2\, (\mu^2 s + yz) = ys\pi + y^3 z = (ys + xz)\,\pi = \pi^2, \\
\lambda^2\mu^8 &= \lambda^2 \quad (\mu^2 s + yz) = ys^2\pi + xzs\pi + y^2 z\pi = (s\pi + y^2 z)\,\pi .
\end{aligned}$$

Consequently, the multiplication of $\overline{\overline{\omega_\tau \phi}} = 0$ by λ^2 will make it possible at once to complete the desired substitution and, therefore, leads to the equation of the surface S in the form

$$S \equiv \pi^2 . (z, s)^{\frac{1}{2}(p+\tau)-3} + \pi . \psi''_{\frac{1}{2}(p+\tau)-1} + y^2 . \psi_{\frac{1}{2}(p+\tau)-1} + x . \psi_{\frac{1}{2}(p+\tau)} = 0 .$$

The order of the surface S is $m' = \frac{1}{2}(p+\tau) + 1$. The first group of terms in the equation as written is of degree two in x, y; the second group is of degree as great as three in x, y if $\frac{1}{2}(p+\tau) - q \geqq 3$ and as great as two if $\frac{1}{2}(p+\tau) - q < 3$; the last two groups are of at least the $[\frac{1}{2}(p+\tau) - q + 1]^{\text{th}}$ degree in x, y; and neither the first nor both the last two groups of terms can be wanting in the equation; while, if $\frac{1}{2}(p+\tau) - q = 0$, the occurrence in $\omega\phi = 0$ of a term or terms of the form $(\lambda, \mu)^{p+\tau-q} . \nu^q$ leads to the occurrence in the equation of S of a term or terms of the form $x . (z, s)^{\frac{1}{2}(p+\tau)}$, and, consequently, the last group of terms must occur whenever $\frac{1}{2}(p+\tau) - q = 0$. Hence, if $\frac{1}{2}(p+\tau) - q \geqq 1$, the surface S contains the double line of Σ twice and has contact of order $\frac{1}{2}(p+\tau) - q - 1$ with the first sheet of Σ all along that line, while the two sheets of S form a cuspidal edge of contact with the first sheet of Σ all along that line when $\frac{1}{2}(p+\tau) - q \geqq 3$. If $\frac{1}{2}(p+\tau) - q = 0$, the surface S contains the double line of Σ once and has contact of the first order with the second sheet of Σ all along that line. In either case, the double line of Σ occurs as a component of the residual intersection $\frac{1}{2}(p+\tau) - q + 1$ times in the first sheet of Σ and twice in the second sheet of Σ, making a total order of $\frac{1}{2}(p+\tau) - q + 3$ for its contribution to the order of the residual intersection. ω_τ introduces τ generators of Σ into the residual intersection, and thus there are already found, as belonging to the residual intersection, straight lines together making up an order of $\frac{1}{2}(p+3\tau) - q + 3$ in that intersection; and, since

$$3\left[\tfrac{1}{2}(p+\tau) + 1\right] - \left[\tfrac{1}{2}(p+3\tau) - q + 3\right] = p + q = m,$$

these lines constitute the entire residual intersection.

The results given above apply to those cases where $\frac{1}{2}(p+\tau) - \alpha_1$ has a value not less than 3; α_1 has been seen to denote the order of the multiplicity of the curve ϕ at that point in the second sheet of Σ where the generator $\mu = 0$ meets the double line, i. e., the number of points of the curve ϕ which lie in the second sheet at that point. This is not a singular point of the surface Σ, and, by a change of coördinates, it is always possible to give such a form to the equation of the curve ϕ that α_1 shall take a lower value, which lower value may, in general, be made to be zero. For this purpose, it is only necessary to choose for the plane $y = 0$ a plane containing no point of the given curve on the double line in the second sheet. In this way all cases where $\frac{1}{2}(p + \tau) - \alpha_1 \leqq 2$ can be referred to those already considered. But for the sake of obtaining some conditions regarding the nature of the surface S, the cases where $\alpha_1 \leqq \frac{1}{2}(p + \tau) - 2$ will now be somewhat further investigated.

Given, then, $\alpha_1 \geqq \frac{1}{2}(p + \tau) - 2$, it is possible, by the methods employed in the general case, to reduce $\overline{\omega_\tau\phi} = 0$ to the form

$$\overline{\overline{\omega_\tau\phi}} \equiv (\mu^2)^{\frac{1}{2}(p+\tau)-\alpha_1} . z^{\alpha_1} . \psi_0 + (\mu^2)^4 . \psi''_{\frac{1}{2}(p+\tau)-4} + (\mu^2)^3 . \psi''_{\frac{1}{2}(p+\tau)-3}$$
$$+ (\mu^2)^2 . \psi''_{\frac{1}{2}(p+\tau)-2} + \frac{\pi}{\lambda} . \psi''_{\frac{1}{2}(p+\tau)-1} + \mu^2 . \psi_{\frac{1}{2}(p+\tau)-1} + \psi_{\frac{1}{2}(p+\tau)} = 0,$$

where $\psi_0 \neq 0$, and hence may be removed by division. There now arise three cases, in each of which the equation of the surface S is readily obtained on the multiplication of the above equation by λ^2 and completing the substitution throughout. These three cases are

1. If $\alpha_1 = \frac{1}{2}(p + \tau) - 2$,
$$S \equiv y\pi . z^{\frac{1}{2}(p+\tau)-2} + \pi . \psi'''_{\frac{1}{2}(p+\tau)-1} + y^2 . \psi_{\frac{1}{2}(p+\tau)-1} + x . \psi_{\frac{1}{2}(p+\tau)} = 0;$$
2. If $\alpha_1 = \frac{1}{2}(p + \tau) - 1$,
$$S \equiv y^2 . z^{\frac{1}{2}(p+\tau)-1} + \pi . \psi'''_{\frac{1}{2}(p+\tau)-1} + y^2 . \psi_{\frac{1}{2}(p+\tau)-1} + x . \psi_{\frac{1}{2}(p+\tau)} = 0;$$

in both these cases the first term and one at least of the last two groups of terms will occur, and the surface S contains the double line of Σ twice if $\frac{1}{2}(p + \tau) - q \geqq 1$.

3. If $\alpha_1 = \frac{1}{2}(p + \tau)$,
$$S \equiv x\left[z^{\frac{1}{2}(p+\tau)} + \psi_{\frac{1}{2}(p+\tau)}\right] + \pi . \psi'''_{\frac{1}{2}(p+\tau)-1} + y^2 . \psi_{\frac{1}{2}(p+\tau)-1} = 0,$$

where the first group of terms is never wanting, and the surface S contains the double line of Σ once. In all three cases $\psi'''_{\frac{1}{2}(p+\tau)-1}$ is of degree as great as unity in x, y; hence, if $\frac{1}{2}(p + \tau) - q \geqq 1$, the surface S contains the double line of Σ twice and has contact of order $\frac{1}{2}(p + \tau) - q - 1$ with the first sheet of Σ all along that line; and, if $\frac{1}{2}(p + \tau) - q = 0$, the surface S contains the double line of Σ once and has contact of the first order with the second sheet of Σ all along that line. Therefore, here, just as in the general case, the residual intersection is entirely made up of the double line of Σ occurring $\frac{1}{2}(p + \tau) - q + 1$

times in the first sheet and twice in the second sheet of Σ, and of generators of Σ to the total order τ introduced by ω_τ.

Since the surface S contains the curve ϕ but once in general, the equation of that surface will not contain Σ, nor the equation of any other second surface through the given curve, as a factor of all its terms; and, since ϕ is by supposition a proper curve and the entire intersection of S with Σ has been found to consist of the curve ϕ and straight lines, it appears that the only cases in which S is an improper surface occur when S is made up of a surface containing the given curve and another surface or surfaces cutting from Σ only straight lines, the double line and generators of Σ. Such supplementary surfaces are made up of planes through the double line of Σ, and, consequently, are represented by equations of the form $ax + by = 0$. Thus S will be, in general, an improper surface only when its equation contains one or more polynomials of the form $ax + by$ as a factor or factors of all its terms. Since these polynomials are each homogeneous in the variables x, y, if the terms of the equation of S be arranged in groups according to the degrees of those terms in x, y, every such group must by itself contain each such polynomial $ax + by$ as a factor; and, since the equation of the surface S is in no case of greater degree than two in x, y, not more than two factors of the required form can ever occur.

If, then, $ax + by$ be a factor of all the terms of the equation of S, the plane $ax + by = 0$ cuts from Σ a generator introduced into the residual intersection by ω_τ, unless $b = 0$; and if any other of the single infinity of generators of Σ be substituted in ω_τ for the generator cut out by the plane in question, the existence of a corresponding factor will, in general, be impossible in the equation of S, and the surface will become in that case a proper surface. To determine in what cases b can have the value zero, let it be supposed that x is a factor of every term in the equation of S; the plane $x = 0$ contains the double line in the second sheet of Σ twice; the residual intersection of S with Σ contains that line in that sheet also twice; if, now, the factor x be removed from the equation of S, leaving the equation of a surface S' of order $\frac{1}{2}(p + \tau)$, the residual intersection of S' with Σ cannot contain the double line in the second sheet of Σ at all; but, if $\frac{1}{2}(p + \tau) - q \geqq 1$, the equation of S being of the second degree in x, y, the equation of S' must be of the first degree in those variables, and hence the double line of Σ must still occur at least once in the residual intersection of S' with Σ; and, if $\frac{1}{2}(p + \tau) - q = 0$, the equation of S' will be of zero degree in x, y, and, consequently, the residual intersection cannot contain the double line of Σ; on the double line in the first sheet of Σ are $p-q$ points of the given curve and τ points where the τ generators in ω_τ meet that line, while on that line in the second sheet of Σ are q points of the curve only; since the curve and the τ generators must make up the entire intersection of S' with Σ, it is evident that either $\tau=0$ or else the τ generators are so chosen and the nature of the curve is such that the combined locus of the curve and the τ generators have a pair or pairs of branches wherever it meets the double line; since this latter condition is not, in general, fulfilled (and may always be avoided by appropriate choice of ω_τ if $\tau \neq 0$), it follows that τ must here have the value zero. But in that case $p=2q$, $m'=\frac{1}{2}(p+\tau)=\frac{1}{2}p$, and the case is that of complete intersection which has been excluded from the cases now under consideration. From another point of view, since the equation of the curve contains a term $\mu^{p-\alpha_1}.\nu^{\alpha_1}$, there must occur in $\omega_\tau\phi=0$ a term of the form $\mu^{p+\tau-\alpha_1}.\nu^{\alpha_1}$ and x can be a factor of all the terms of the equation of S only if here $p+\tau-\alpha_1 \leqq \alpha_1$; but this condition can be satisfied only when $p+\tau=2\alpha_1=2q$ (since $\alpha_1=q$), and hence, as explained on page 189, the curve ϕ must have all its branches in the second sheet in the neighborhood of the double line meet that line at the point where the generator $\mu=0$ meets it; in other words, the point in the second sheet where the generator $\mu=0$ meets the double line is a q-tuple point of the curve ϕ in that sheet; and not only must every branch of the curve ϕ in the first sheet of Σ in the neighborhood of the double line pass through this same point, making it $(p-q)$-tuple in the first sheet, and hence a p-tuple point on the curve ϕ as a whole, but also must ω_τ have been made up entirely of the generator $\mu=0$ occurring τ times, which, in general, will not be the case. Hence, the cases of complete intersection being excepted, the equation of the surface S will contain the factor x only when ω_τ has been chosen in a very special form, i. e., when $\omega_\tau=\mu^\tau$ or $C.\mu^\tau$.

It has been seen that, when $\frac{1}{2}(p+\tau)-\alpha_1 \geq 3$, the equation of S has the form

$$S \equiv \pi^2.(z,\ s)^{\frac{1}{2}(p+\tau)-3}+\pi.\psi'''_{\frac{1}{2}(p+\tau)-1}+y^2.\psi_{\frac{1}{2}(p+\tau)-1}+x.\psi_{\frac{1}{2}(p+\tau)}=0.$$

If $\frac{1}{2}(p+\tau)-q \geqq 3$, the only terms occurring here in the second degree in x, y are contained in the first group and involve those variables in the form $(xs+yz)^2$; hence, no factor of the form $ax+by$ can occur in this case and S is a proper surface. If $\frac{1}{2}(p+\tau)-q=2$, terms of the second degree in x, y occur only in the first two groups of terms and all have the factor π; therefore, at most, a single factor $ax + by$ can here be found; if such a factor does occur, the plane $ax + by = 0$ cuts from Σ a generator contained in ω_τ, and the substitution of any other generator in that expression in place of the one there occurring will, in general, lead to a proper surface S. If $\frac{1}{2}(p + \tau) - q = 1$, terms of the second degree in x, y can occur in every group above, and if one or, at most, two factors of the given form occur, other generators may be selected for replacing in ω_τ the one or two cut out by the plane or planes in question and a proper surface S will, in general, be thus obtained. If, finally, $\frac{1}{2}(p + \tau) - q = 0$, the terms of the first degree in x, y all occur in the last group and have x as the only possible factor of the required form; but it has been already seen that a proper surface S can in this case always be found.

There still remain the cases where $\frac{1}{2}(p + \tau) - \alpha_1 \leqq 2$, but these can be referred, by a change of coördinates, to those already considered. It is clear, however, from the forms of the equations of S, as given on page 193, that the results obtained above find immediate application here; and, since $\alpha_1 \leqq q$, only those cases where $\frac{1}{2}(p + \tau) - q \leqq 2$ demand consideration. The equations of S, as well as the fact that the point cut from the double line in the second sheet of Σ by the plane $y = 0$ is a point of multiplicity of order $\alpha_1 \geqq q - 2$ on the curve ϕ in that sheet, suggest that in the cases in question the methods followed, without change of coördinates, may lead to the occurrence of the plane $y = 0$ as a component of the surface S. An upper limit for θ_2, which shall render this impossible, is readily obtained thus: The term of the equation of ϕ, in which λ is wanting and ν appears to the power α_1, when multiplied by that term of ω_τ, which is of the lowest degree in μ, will take the following form for substitution:

$$C_1\mu^{p-\alpha_1}\nu^{\alpha_1}.\, C_2\lambda^{\tau-\theta_2}\mu^{\theta_2} = C'\,(\mu\nu)^{p-\alpha_1+\theta_2}.\,(\lambda\nu)^{2\alpha_1-p-\theta_2}.\,\lambda^{\tau+p-2\alpha_1},$$

if $\theta_2 \leqq 2\alpha_1 - p$; and this condition for θ_2 is, then, a sufficient one to insure that the equation of the surface S be not reducible by the factor y. This condition, in the three most unfavorable cases, takes the forms

1. $\theta_2 \leqq \tau$, when $\frac{1}{2}(p + \tau) = q$.
2. $\theta_2 \leqq \tau - 2$, when $\frac{1}{2}(p + \tau) = q + 1$.
3. $\theta_2 \leqq \tau - 4$, when $\frac{1}{2}(p + \tau) = q + 2$.

Since the generator $\mu = 0$ is in no way singular on Σ, whatever is true for it may, by a change of coördinates, be made to apply, in general, to any other generator in the first sheet of Σ. And it has been seen that S is a proper surface at once, save only in those cases where ω_τ contains a certain generator or generators, having a particular relation to the curve in question, more than a certain easily ascertainable number of times. From the single infinity of generators on Σ it will always be possible to make up ω_τ in such a way that these exceptional cases may, from the outset, be avoided.

It is now clear that in every case, in general, where τ is so taken that $p + \tau \geqq 2q$, and even, while in addition $\theta_1 = 0$ and $\theta_2 \leqq \frac{1}{2}(p + \tau)$, then can the method given be made to determine the equation of a proper surface S of any desired order $m' = \frac{1}{2}(p + \tau) + 1 \geqq \frac{1}{2}p + 1$. Since a surface S of order $\frac{1}{2}(p + \tau) + 1$ cannot contain a line of greater multiplicity than $\frac{1}{2}(p + \tau)$ without breaking up into surfaces of lower orders, it is evident that S must have contact with Σ of order at least as great as one all along the generator $\mu = 0$ whenever $\theta_2 = \frac{1}{2}(p + \tau) + 1$. Since the generators together make up a total order τ in the residual intersection, there will be contact between S and Σ all along one or more of the generators of Σ in question whenever less than τ different generators enter in ω_τ, unless the occurrence of one or more of the generators in question as multiple lines of S accounts for the entire number of lines of intersection by which τ exceeds the number of different generators involved in ω_τ. In general, then, subject to the conditions already given, ω_τ may be chosen to contain any τ generators whatever lying in the first sheet of Σ, or any lesser number of generators in that sheet to become, in part at least, multiple lines of S, or lines of contact between the two surfaces S and Σ; but, in every case, the total order in the intersection must be τ, no generator can be of multiplicity of order greater than $\frac{1}{2}(p + \tau)$ on S, and any generator occurring more than $\frac{1}{2}(p + \tau)$ times in the intersection is a line of contact of S with Σ.

If it be desired so to choose ω_τ that a reduction by one or two degrees in the order of the surface S be obtained, it is evident that τ must be so chosen as to make $\frac{1}{2}(p + \tau) - q = 0$ or 2 for a reduction by unity, or $\frac{1}{2}(p + \tau) - q = 1$ for a reduction by unity or two. And, further, it must be possible to choose ω_τ in such a way that at every point where a branch of the curve ϕ meets the double line, shall occur a pair or pairs of branches of the locus $\omega_\tau\phi = 0$, if the reduction is to be by two degrees, or if $\frac{1}{2}(p + \tau) - q = 0$ and the reduction is to be by unity. Sufficient conditions for the reduction of the order of the surface S in the special cases, where alone such reduction is possible, will not be considered further here.

The cases where λ occurs as a factor of all the terms of ω_τ, and where, consequently, $\theta_1 > 0$, can all be solved in a manner entirely similar to that already given for the cases where $\theta_1 = 0$. The term of highest degree in μ arising in $\omega_\tau\phi = 0$ from any term $C.\lambda^{p-\alpha-\beta}.\mu^\alpha.\nu^\beta$ of $\phi = 0$ is of the form $C'.\lambda^{p-\alpha-\beta+\theta_1}.\mu^{\alpha+\tau-\theta_1}.\nu^\beta$, which, when factored in the manner prescribed for substitution, becomes

$$C.(\mu\nu)^\beta.(\lambda\mu)^{p-\alpha-\beta+\theta_1}.(\mu^2)^{\frac{1}{2}(\tau-p)+\alpha-\theta_1}, \text{ if } \theta_1 \leqq \tfrac{1}{2}(\tau-p)+\alpha.$$

Since α must take the value $\alpha \geqq p-q$ in some term of $\phi = 0$, a sufficient condition that some term of $\overline{\omega_\tau\phi} = 0$ in this case contain $(\mu^2)^\epsilon$, where $\epsilon \geqq 2$, is that $\theta_1 \leqq \frac{1}{2}(p+\tau)-q-1$; and, if $\theta_1 = \frac{1}{2}(\tau-p)+\alpha_2-1$, where α_2 is the largest value assumed by α in the equation of ϕ, then will μ^2 occur in $\overline{\omega_\tau\phi} = 0$ while μ^4 will not occur in that equation. Hence, the condition $\theta_1 \geqq \frac{1}{2}(p+\tau)$ causes $\overline{\omega_\tau\phi} = 0$ to be at once the equation of the surface S, which may be a proper surface if $\alpha_2 = p$, but will contain as a component the plane $x = 0$ if $\alpha_2 \leqq p-1$. It will now be supposed that θ_1 is allowed to have any value consistent with the newly obtained condition, viz., that

$$\theta_1 \leqq \tfrac{1}{2}(p+\tau)-q-1.$$

This condition involves that $p+\tau-\theta_1 \geqq 2q$, and hence the equation $\overline{\omega_{\tau-\theta_1}\phi} = 0$ can, when $p+\tau-\theta_1$ is even, like the equation $\overline{\omega_\tau\phi} = 0$ on page 188, be written in the partially substituted form thus:

$$\begin{aligned}\overline{\omega_{\tau-\theta_1}\phi} \equiv (\mu^2)^{\frac{1}{2}(p+\tau-\theta_1)}.\psi_0 + (\mu^2)^{\frac{1}{2}(p+\tau-\theta_1)-1}.\psi_1 + \dots\\ + (\mu^2)^{\frac{1}{2}(p+\tau-\theta_1)-\kappa}.\psi_\kappa + \dots + \mu^2.\psi_{\frac{1}{2}(p+\tau-\theta_1)-1} + \psi_{\frac{1}{2}(p+\tau-\theta_1)} = 0.\end{aligned}$$

If $p+\tau-\theta_1$ be odd, the equation $\overline{\omega_{\tau-\theta_1}\phi} = 0$ will take the same form, with the exception that an odd instead of an even power of μ is a factor of every group of terms save the last, and that the last group of terms has the form $\lambda.\overline{\psi}_{\frac{1}{2}(p+\tau-\theta_1-1)} + \mu.\overline{\overline{\psi}}_{\frac{1}{2}(p+\tau-\theta_1-1)}$; but in either case multiplication by λ^{θ_1+2}, after the method used when $\theta_1 = 0$, leads to the equation of S in the form

$$S \equiv [\pi^2.(z, s)^{\frac{1}{2}(p+\tau)-\theta_1-3} + \pi.\psi'''_{\frac{1}{2}(p+\tau)-0_1-1}].y^{\theta_1} + \overline{\psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 \geqq 3$; but in the form

$$S \equiv [y\pi . z^{\frac{1}{2}(p+\tau)-\theta_1-2} + \pi . \psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}] \cdot y^{\theta_1} + \overline{\Psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 = 2$; and in the form

$$S \equiv [y^2 . z^{\frac{1}{2}(p+\tau)-\theta_1-1} + \pi . \psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}] \cdot y^{\theta_1} + \overline{\Psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 = 1$; and, finally, in the form

$$S \equiv [x . z^{\frac{1}{2}(p+\tau)-\theta_1} + \pi . \psi'''_{\frac{1}{2}(p+\tau)-\theta_1-1}] \cdot y^{\theta_1} + \overline{\Psi}_{\frac{1}{2}(p+\tau)+1} = 0,$$

if $\frac{1}{2}(p+\tau)-\theta_1-\alpha_1 = 0$.
Just as in the case where $\theta_1 = 0$, so here it can be shown that neither the first nor the last group of terms can fail to occur in the equation of the surface S; and, when $\frac{1}{2}(p+\tau)-q \geqq 1$, the surface S contains the double line of Σ θ_1+2 times, and has contact of order $\frac{1}{2}(p+\tau)-q-\theta_1-1$ with the first sheet of Σ all along that line; similarly, if $\frac{1}{2}(p+\tau)-q=0$, the surface S contains the double line of Σ θ_1+1 times and has contact of the first order with the second sheet of Σ all along that line. Consequently, the residual intersection is here entirely made up in every case, in general, of the double line of Σ occurring $\frac{1}{2}(p+\tau)-q+1$ times in the first sheet and θ_1+2 times in the second sheet of Σ, and of generators of Σ to the total order $\tau-\theta_1$ introduced by $\omega_{\tau-\theta_1}$. Hence, the generator of Σ lying coincident with the double line in the second sheet enters like any other generator into the residual intersection, so long, at least, as θ_1 has any value within the limits assigned; and thus this case can be included in the general one and the results of the considerations of the nature of the surface S, and of the resultant residual intersection in particular, may be extended so as to include all cases, without making any distinction between the generator in the second sheet of Σ and any other generator of that surface, provided only that θ_1 and θ_2 do not exceed the limits $\frac{1}{2}(p+\tau)-q-1$ and $\frac{1}{2}(p+\tau)+1$ respectively. Therefore, there may be stated the following

THEOREM: *A proper surface S, having for its partial intersection with Σ any given proper curve of order m on Σ, can be found of any desired order $m' = \frac{1}{2}(p+\tau)+1$, where $\tau \geqq 0$, and $p+\tau \geqq 2q$ and even, whose residual intersection with Σ shall consist entirely of the double line of Σ occurring $\frac{1}{2}(p+\tau)-q+1$ times in the first sheet of Σ and twice in the second sheet of Σ, together with arbitrary generators of Σ occurring in such manner as to give a total order of τ in the intersec-*

tion, while no particular generator occurs more than $\frac{1}{2}(p+\tau)-q-1$ *times among such arbitrary generators.*

In general, ω_τ must have $\theta_1 \leqq \frac{1}{2}(p+\tau)-q-1$, and, for values of θ_1 which are greater than zero, the generator in the second sheet of Σ must be regarded as occurring θ_1 times in the residual intersection, in addition to the two times given in the theorem. That this line always occurs twice in the residual intersection is the natural result of the multiplication of $\overline{\overline{\omega_\tau \phi}} = 0$ in every case by λ^2. Also must ω_τ have $\theta_2 \leqq \frac{1}{2}(p+\tau)-1$, and, if ω_τ introduce only generators in the first sheet of Σ which do not meet the given curve on the double line of Σ, the particular method of substitution given will lead at once to a proper surface S. Clearly, by varying the form of ω_τ, an infinite number of such surfaces S of any order $m' = \frac{1}{2}(p+\tau)+1$ can be found for any given curve ϕ.

WILLIAMS COLLEGE, *February* 1, 1900.

(*To be continued.*)